coupled mode theory at such a large θ value fails due to the approximation. However, strong coupling phenomenon is extrapolated from Fig. 4, which spreads out the stop bandwidth around point P. The resonant frequency of MSSW ($\omega \simeq \omega_h + \gamma\mu_0 M/2$) may be also included within stopband. Thus coupled power is absorbed by the resonance behavior and sharp cut-off of more than 30 dB may be observed.

IV. Conclusion

We have proposed a band rejection filter which uses the coupling between TEM and MSSW modes in the YIG film microstrip line. Experiments have been performed by using 40 μm thick YIG film, 400 μm thick GGG and 0.7 mm width strip, and for various dc magnetic field directions. Sharp notch characteristics of more than 30 dB have been observed at X band, and results are phenomenologically explained with a coupled mode theory.

These characteristics are very useful for magnetostatic wave application to band rejection filters at microwave frequency.

Acknowledgment

We thank Murata Mfg. Co. Ltd for technical assistance.

References

[1] J. D. Adam, "Analog signal processing with microwave magnetics," *Proc. IEEE.*, vol. 76, no. 2, pp. 159-170 Feb. 1988.
[2] W. Ishak and K. W. Chang, "Tunable microwave resonators using magnetostatic wave in YIG films," *IEEE Trans. Microwave Theory Tech.*, vol. 34, no. 12, pp. 1383-1393, Dec. 1986.
[3] A. Yariv, "Coupled-mode theory for guided-wave optics," *IEEE, J. Quantum Electron.*, vol. 9, no. 9, pp. 919-933, Sept. 1973.
[4] A. Yariv, *Optical Electronics*. New York: CBS College Publishing, 1985.
[5] P. R. Emtage, "Interaction of magnetostatic waves with a current," *J. of Appl. Phys.* vol. 49, no. 8, pp. 4475-4484, Aug. 1978.
[6] S. N. Bajpai, "Insertion loss of electronically variable magnetostetic wave delay lines," *IEEE. Trans. Microwave Theory Tech.*, vol. 37, no. 10, pp. 1529-1535, Oct. 1989.
[7] B. Lax and K. J. Button, *Microwave Ferrites and Ferrimagnetics*. New York: McGraw Hill, 1962.
[8] T. J. Gerson and J. S. Nadan, "Surface electromagnetic modes of ferrite slab," *IEEE. Trans. Microwave Theory Tech.*, vol. 22, no. 8, pp. 757-762, Aug. 1974.
[9] M. Tsutsumi and T. Asahara, "Microstrip lines using Yttrium Iron Garnet film," *IEEE. Trans. Microwave Theory Tech.*, vol. 38, no. 10, pp. 1461-1467, Oct. 1990.

Variational Method for the Analysis of Lossless Bi-Isotropic (Nonreciprocal chiral) Waveguides

Ismo V. Lindell

Abstract—**Equations are derived for the longitudinal fields of a propagating mode in the most general straight open waveguide structure made from the most general lossless linear material whose parameters are independent of its orientation. This material, also called bi-isotropic, has the important chiral medium as the reciprocal special case. Self-adjointness of the differential operator with respect to the hermitian inner product is confirmed. Applying the theory of nonstandard eigenvalue problems, a variational expression is derived for the solution of the waveguide problem. A procedure for its application is discussed.**

Manuscript received April 24, 1991; revised September 25, 1991.
The author is with the Electromagnetics Laboratory, Helsinki University of Technology, Otakaari 5A, Espoo, Finland 02150.
IEEE Log Number 9104763.

I. Introduction

The chiral medium has raised considerable theoretical interest in recent years because of its unique properties. In fact, offering an extra parameter, it gives the possibility of satisfying conditions beyond those of isotropic media. Among suggested applications we may mention realization of reflectionless surfaces, for which there exist numerous patents [1], creating polarization rotating microwave devices without ferrites [2], and interesting antennas, like closely packed microstrip antennas with less coupling between the elements [3]. Also there exists a monograph on electromagnetics in chiral media [4].

The "chiral medium" treated in the literature has mostly been reciprocal. However, accepting nonreciprocity, we have one more parameter to deal with and the medium can be called nonreciprocal chiral or, more generally, bi-isotropic. Such a medium without chirality was introduced by Tellegen in 1948 to realize a new circuit element called the gyrator [5]. Wave propagation in a bi-isotropic medium was recently studied by J. C. Monzon [6]. The medium equations can be written in the form [7]

$$\boldsymbol{D} = \epsilon \boldsymbol{E} + \xi \boldsymbol{H}, \tag{1}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} + \zeta \boldsymbol{E}, \tag{2}$$

with

$$\xi = (\chi - j\kappa)\sqrt{\mu_o \epsilon_o}, \quad \zeta = (\chi + j\kappa)\sqrt{\mu_o \epsilon_o}. \tag{3}$$

For lossless media, the parameters κ and χ together with ϵ and μ are real, whence $\xi = \zeta^*$, which case is assumed here. The chirality parameter κ gives the rate of polarization rotation of a linearly polarized plane wave, relative to the rate of phase change of the wave in propagation. The Tellegen parameter χ is proportional to the polarization rotation of a plane wave in reflection from a discontinuity [7].

In the present paper, the electromagnetic problem of wave propagation along a general straight open waveguide of bi-isotropic material is formulated in terms of a variational method. A similar method has been previously derived and applied for dielectric and corrugated waveguides and labeled as 'a variational method for nonstandard eigenvalue problems' [8]–[11]. The nonstandard, or nonlinear, eigenvalue problem is one which can be written in the general operator from $L(\lambda)f = 0$, where the eigenvalue parameter λ is not necessarily a linear coefficient. The result of this way of thinking [8] is that any parameter of the problem can be taken as the eigenvalue parameter λ and if it can be solved analytically from a functional equation, what results is a stationary functional for that parameter. This is also the procedure followed in this paper for the bi-isotropic problem.

II. Theory

The bi-isotropic medium considered here is homogeneous along one space direction labeled with the z coordinate and a function of the transverse position vector $\boldsymbol{\rho} = \boldsymbol{r} - \boldsymbol{u}_z(\boldsymbol{u}_z \cdot \boldsymbol{r})$. Let us asume that the structure is concentrated close to the z axis and the parameters approach those of the free space for $\rho = |\boldsymbol{\rho}| \to \infty$. Also let us only consider waveguide-mode solutions whose fields decay radially outwards more rapidly than $1/\sqrt{\rho}$. This boundary condition at infinity could also be replaced by suitable conditions at a finite distance to analyze the closed waveguide.

A. Basic Equations

To simplify the notation in the beginning, let use write

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix}, \quad \mathbf{G}(\mathbf{r}) = \begin{pmatrix} \mathbf{D}(\mathbf{r}) \\ \mathbf{B}(\mathbf{r}) \end{pmatrix} \tag{4}$$

for the two pairs of field vectors. Assuming a propagating wave solution,

$$\mathbf{F}(\mathbf{r}) = [\mathbf{u}_z f(\boldsymbol{\rho}) + \mathbf{f}(\boldsymbol{\rho})] e^{-j\beta z}, \tag{5}$$

$$\mathbf{G}(\mathbf{r}) = [\mathbf{u}_z g(\boldsymbol{\rho}) + \mathbf{g}(\boldsymbol{\rho})] e^{-j\beta z}, \tag{6}$$

$$f(\boldsymbol{\rho}) = \begin{pmatrix} e(\boldsymbol{\rho}) \\ h(\boldsymbol{\rho}) \end{pmatrix}, \quad g(\boldsymbol{\rho}) = \begin{pmatrix} d(\boldsymbol{\rho}) \\ b(\boldsymbol{\rho}) \end{pmatrix},$$
$$\mathbf{f}(\boldsymbol{\rho}) = \begin{pmatrix} \mathbf{e}(\boldsymbol{\rho}) \\ \mathbf{h}(\boldsymbol{\rho}) \end{pmatrix}, \quad \mathbf{g}(\boldsymbol{\rho}) = \begin{pmatrix} \mathbf{d}(\boldsymbol{\rho}) \\ \mathbf{b}(\boldsymbol{\rho}) \end{pmatrix}, \tag{7}$$

where f, g involve the longitudinal (z directed) field components and $\mathbf{f}$, $\mathbf{g}$, the corresponding transverse components satisfying $\mathbf{u}_z \cdot \mathbf{f} = 0$, $\mathbf{u}_z \cdot \mathbf{g} = 0$. The Maxwell curl equations in a sourceless region can be written in the form

$$\nabla \times \mathbf{F}(\mathbf{r}) = j\omega J \mathbf{G}(\mathbf{r}) = j\omega J M(\boldsymbol{\rho}) \mathbf{F}(\mathbf{r}). \tag{8}$$

The medium parameter matrix is defined as

$$M = \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix} = \begin{pmatrix} \epsilon & (\chi - j\kappa)\sqrt{\mu_o \epsilon_o} \\ (\chi + j\kappa)\sqrt{\mu_o \epsilon_o} & \mu \end{pmatrix} \tag{9}$$

and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

$M(\boldsymbol{\rho})$ is a function of the transverse position vector $\boldsymbol{\rho}$. For a lossless medium, which is assumed here, the parameters ϵ, μ, χ and κ have real values.

B. Equation for the Longitudinal Fields

Let us now eliminate the transverse field components to obtain equations for the longitudinal components alone. For this, the Maxwell equations (8) are written as

$$-j\beta \mathbf{u}_z \times \mathbf{f}(\boldsymbol{\rho}) + \nabla \times \mathbf{f}(\boldsymbol{\rho}) - \mathbf{u}_z \times \nabla f(\boldsymbol{\rho})$$
$$= j\omega \mathbf{u}_z J g(\boldsymbol{\rho}) + j\omega J \mathbf{g}(\boldsymbol{\rho}). \tag{11}$$

When the components parallel and transverse to $\mathbf{u}_z$ are separated, we have the two equations

$$\mathbf{u}_z \cdot \nabla \times \mathbf{f}(\boldsymbol{\rho}) = j\omega J g(\boldsymbol{\rho}) = j\omega J M f(\boldsymbol{\rho}), \tag{12}$$

$$-j\beta \mathbf{u}_z \times \mathbf{f}(\boldsymbol{\rho}) - \mathbf{u}_z \times \nabla f(\boldsymbol{\rho}) = j\omega J \mathbf{g}(\boldsymbol{\rho}) = j\omega J M \mathbf{f}(\boldsymbol{\rho}). \tag{13}$$

Let us solve for $\mathbf{f}$ from (13) in terms of f, and substitute this in (12). Writing (13) in the form

$$\omega J M \mathbf{f} + \beta \mathbf{u}_z \times \mathbf{f} = (\omega J M \bar{\bar{I}}_t + \beta I \mathbf{u}_z \times \bar{\bar{I}}) \cdot \mathbf{f} = j\mathbf{u}_z \times \nabla f, \tag{14}$$

where I denotes the unit 2×2 matrix and $\bar{\bar{I}}_t = \mathbf{u}_x \mathbf{u}_x + \mathbf{u}_y \mathbf{u}_y$ is the transverse unit dyadic, we can solve for $\mathbf{f}$ by finding the inverse of the dyadic matrix in brackets. Better still, noting that $J^2 = -I$, (14) can be written as

$$\bar{\bar{D}} \cdot \mathbf{f} = -jJ\mathbf{u}_z \times \nabla f, \tag{15}$$

$$\bar{\bar{D}} = \omega M \bar{\bar{I}}_t - \beta J \mathbf{u}_z \times \bar{\bar{I}}$$
$$= \begin{pmatrix} \omega\epsilon \bar{\bar{I}}_t & \omega\xi \bar{\bar{I}}_t + \beta \mathbf{u}_z \times \bar{\bar{I}} \\ \omega\zeta \bar{\bar{I}}_t - \beta \mathbf{u}_z \times \bar{\bar{I}} & \omega\mu \bar{\bar{I}}_t \end{pmatrix}. \tag{16}$$

Inverting this dyadic matrix is an easy task, because all the dyadic elements commute. In fact, for a matrix of dyadics $\bar{\bar{A}}$, $\bar{\bar{B}}$, $\bar{\bar{C}}$ and $\bar{\bar{D}}$ which all commute: $\bar{\bar{A}} \cdot \bar{\bar{B}} = \bar{\bar{B}} \cdot \bar{\bar{A}}$ etc., we have the identity

$$\begin{pmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ \bar{\bar{C}} & \bar{\bar{D}} \end{pmatrix} \cdot \begin{pmatrix} \bar{\bar{D}} & -\bar{\bar{B}} \\ -\bar{\bar{C}} & \bar{\bar{A}} \end{pmatrix} = (\bar{\bar{A}} \cdot \bar{\bar{D}} - \bar{\bar{B}} \cdot \bar{\bar{C}}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{17}$$

whence the inverse (left or right) can be written in a form resembling that of a matrix with scalar elements:

$$\begin{pmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ \bar{\bar{C}} & \bar{\bar{D}} \end{pmatrix}^{-1} = (\bar{\bar{A}} \cdot \bar{\bar{D}} - \bar{\bar{B}} \cdot \bar{\bar{C}})^{-1} \cdot \begin{pmatrix} \bar{\bar{D}} & -\bar{\bar{B}} \\ -\bar{\bar{C}} & \bar{\bar{A}} \end{pmatrix}. \tag{18}$$

For noncommuting dyadics this is, however, not true.

The inverse of (16) can thus be written in the following form:

$$\bar{\bar{D}}^{-1} = \bar{\bar{K}}^{-1} \cdot \begin{pmatrix} \omega\mu \bar{\bar{I}}_t & -\omega\xi \bar{\bar{I}}_t - \beta \mathbf{u}_z \times \bar{\bar{I}} \\ -\omega\zeta \bar{\bar{I}}_t + \beta \mathbf{u}_z \times \bar{\bar{I}} & \omega\epsilon \bar{\bar{I}}_t \end{pmatrix}, \tag{19}$$

$$\bar{\bar{K}} = k_c^2 \bar{\bar{I}}_t - 2jk_o \kappa \beta \mathbf{u}_z \times \bar{\bar{I}}, \quad \bar{\bar{K}}^{-1} = \frac{1}{k_c^4 - (2k_o \kappa \beta)^2} \bar{\bar{K}}^T, \tag{20}$$

$$k_c^2 = k^2 - \beta^2 - k_o^2(\chi^2 + \kappa^2) = k_o^2(n^2 - \chi^2 - \kappa^2) - \beta^2, \tag{21}$$

where n denotes the refraction factor $\sqrt{\mu_r \epsilon_r}$. For isotropic guides with $\chi = \kappa = 0$, k_c plays the role of cutoff wavenumber, because $k = k_c$ implies $\beta = 0$.

Looking at the expression (19), we can further write

$$\bar{\bar{D}}^{-1} = -\bar{\bar{K}}^{-1} \cdot (\omega J M^T J \bar{\bar{I}}_t - \beta J \mathbf{u}_z \times \bar{\bar{I}}). \tag{22}$$

The transverse fields can now be solved in terms of the longitudinal fields. From (15) and (19) we have

$$\mathbf{f} = -j\bar{\bar{D}}^{-1} J \cdot \mathbf{u}_z \times \nabla f. \tag{23}$$

Substituting this in (12) leaves us with an equation for the longitudinal fields alone, which can be written in the form

$$L(\nabla) f = \nabla \cdot [\bar{\bar{K}}^{-1} \cdot (\omega M^T \bar{\bar{I}}_t + \beta J \mathbf{u}_z \times \bar{\bar{I}}) \cdot \nabla f] + \omega M f = 0. \tag{24}$$

This equation serves also as a definition of the differential operator matrix $L(\nabla)$. Written in terms of the longitudinal electric and magnetic field components, (24) has the form of two scalar equations

$$\nabla \cdot [\bar{\bar{K}}^{-1} \cdot (\omega\epsilon \nabla e + \omega\zeta \nabla h - \beta \mathbf{u}_z \times \nabla h)] + \omega(\epsilon e + \xi h) = 0, \tag{25}$$

$$\nabla \cdot [\bar{\bar{K}}^{-1} \cdot (\omega\mu \nabla h + \omega\xi \nabla e + \beta \mathbf{u}_z \times \nabla e)] + \omega(\mu h + \zeta e) = 0. \tag{26}$$

As a check, we see that for an isotropic medium with $\chi = \kappa = 0$, we have $\bar{\bar{K}} = k_c^2 \bar{\bar{I}}_t = (k^2 - \beta^2)\bar{\bar{I}}_t$, whence (25), (26) reduce to

$$\nabla \cdot [k_c^{-2} \cdot (\omega\epsilon\nabla e - \beta \boldsymbol{u}_z \times \nabla h)] + \omega\epsilon e = 0, \tag{27}$$

$$\nabla \cdot [k_c^{-2} \cdot (\omega\mu\nabla h + \beta \boldsymbol{u}_z \times \nabla e)] + \omega\mu h = 0, \tag{28}$$

which coincide with previously derived equations in [10]. For homogeneous media we have the simple equations $\nabla^2 e + k_c^2 e = 0$ and $\nabla^2 h + k_c^2 h = 0$.

C. The Variational Method

To find a variational expression for the waveguide problem, let us first define the following hermitian inner product for two pairs of scalar field functions $f_1 = (e_1 h_1)$ and $f_2 = (e_2 h_2)$:

$$(f_1, f_2) = \int_S f_1^* f_2 \, dS = \int_S (e_1^* e_2 + h_1^* h_2) \, dS. \tag{29}$$

The range of integration S extends over the whole xy plane.

Now it can be shown that the operator $L(\nabla)$, defined in (24), is self adjoint with respect to this inner product if the medium is lossless. In fact, if we write

$$(f_1, L(\nabla) f_2) = \int_S f_1^* [\nabla \cdot (\bar{\bar{K}}^{-1} \cdot (\omega M^T \nabla f_2 + \beta J \boldsymbol{u}_z \times \nabla f_2)) + \omega M f_2] \, dS, \tag{30}$$

we can show that each term of the operator has a corresponding self-adjoint term. Let us study each of the three terms in the integral individually. Applying the Gaussian integral theorem, according to which the divergence terms integrated give line integrals in the infinity, we can drop these because of the assumed behavior of the fields in the infinity. Thus, for the three terms we can write

$$\begin{aligned}\omega \int_S f_1^* \nabla \cdot [\bar{\bar{K}}^{-1} \cdot M^T \nabla f_2] \, dS &= -\omega \int_S [M \bar{\bar{K}}^{-1T} \cdot \nabla f_1^*] \cdot \nabla f_2 \, dS \\ &= \omega \int_S \nabla \cdot [\bar{\bar{K}}^{-1T*} \cdot M^{T*} \nabla f_1]^* f_2 \, dS\end{aligned} \tag{31}$$

$$\begin{aligned}\beta \int_S f_1^* \nabla \cdot [\bar{\bar{K}}^{-1} \cdot J(\boldsymbol{u}_z \times \bar{\bar{I}}) \cdot \nabla f_2] \, dS &= -\beta \int_S [J^T (\boldsymbol{u}_z \times \bar{\bar{I}})^T \cdot \bar{\bar{K}}^{-1T} \cdot \nabla f_1^*] \cdot \nabla f_2 \, dS \\ &= \beta \int_S [\nabla \cdot (J \bar{\bar{K}}^{-1T*} \cdot \boldsymbol{u}_z \times \nabla f_1]^* f_2 \, dS,\end{aligned} \tag{32}$$

$$\omega \int_S f_1^* (M f_2) \, dS = \omega \int_S (M^{T*} f_1)^* f_2 \, dS. \tag{33}$$

In fact, for a lossless medium with real parameters ϵ, μ, χ and κ, the dyadic $\bar{\bar{K}}$ and, hence, $\bar{\bar{K}}^{-1}$, is hermitian satisfying $\bar{\bar{K}}^{-1T*} = \bar{\bar{K}}^{-1}$. Also, the matrix M is hermitian: $M^{T*} = M$. Thus, in this case, in all integrals (31)–(33), the operators are self adjoint and, hence, the whole operator $L(\nabla)$ is self adjoint.

The variational expression can be obtained from the functional equation [8]

$$(f, L(\nabla) f) = 0, \tag{34}$$

which in this case can be written as

$$\int_S f^* [\nabla \cdot (\bar{\bar{K}}^{-1} \cdot (\omega M^T \nabla f + \beta J \boldsymbol{u}_z \times \nabla f)) + \omega M f] \, dS = 0, \tag{35}$$

or, what is equivalent, after partial integrations, as

$$\omega \int_S \nabla f^* \cdot \bar{\bar{K}}^{-1} \cdot M^T \nabla f \, dS + \beta \int_S \nabla f^* \cdot \bar{\bar{K}}^{-1} \cdot J \boldsymbol{u}_z \times \nabla f \, dS = \omega \int_S f^* M f \, dS. \tag{36}$$

It was shown in [8] that, if the functional equation can be solved for any independent parameter of the problem in terms of the field f and other parameters, the expression obtained is stationary for the true solution of the problem. The solvable parameter is called the (nonstandard) eigenvalue parameter. Thus, at this point we must find a parameter which can be solved from (36) in algebraic form. Taking the parameters ω and β as independent ones shows us that neither of them can be solved, because of the complicated relation through the dyadic $\bar{\bar{K}}^{-1}$. However, following the method of reference [10], we can change to two new parameters ω^2 and $v_p = \omega/\beta$, treat them as independent ones and actually solve (36) for the parameter ω^2.

To make this, let us write the dyadic $\bar{\bar{K}}$ as

$$\bar{\bar{K}}(\omega, \beta) = \omega^2 \bar{\bar{L}}(v_p), \tag{37}$$

with

$$\bar{\bar{L}}(v_p) = \left(\frac{1}{c^2}(n^2 - \chi^2 - \kappa^2) - \frac{1}{v_p^2}\right) \bar{\bar{I}}_t - 2j \frac{\kappa}{c v_p} \boldsymbol{u}_z \times \bar{\bar{I}}, \tag{38}$$

where $c = 1/\sqrt{\mu_o \epsilon_o}$ is the speed of light in vacuum. The functional equation (36) turns out now to be solvable for ω^2 in the form

$$\omega^2 = J(v_p; f) = \frac{\int_S \nabla f^* \cdot \bar{\bar{L}}^{-1} \cdot \left(M^T \nabla f \, dS + \frac{1}{v_p} J \boldsymbol{u}_z \times \nabla f\right) dS}{\int_S f^* M f \, dS}. \tag{39}$$

Substituting the longitudinal electric and magnetic fields e, h in the combined field matrix f in (39), the resulting expression for the functional reads

$$\omega^2 = J(v_p; e, h) = \frac{\int_S \left(\nabla e^* \cdot \bar{\bar{L}}^{-1} \cdot \left[\epsilon \nabla e + \zeta \nabla h - \frac{1}{v_p} \boldsymbol{u}_z \times \nabla h\right] + \nabla h^* \cdot \bar{\bar{L}}^{-1} \cdot \left[\mu \nabla h + \xi \nabla e + \frac{1}{v_p} \boldsymbol{u}_z \times \nabla e\right]\right) dS}{\int_S [\epsilon |e|^2 + \mu |h|^2 + 2\Re\{\zeta e h^*\} \, dS}. \tag{40}$$

Let us check the functionals (39) and (40) by studying their special cases for the isotropic guide, i.e., for $\xi = \zeta = 0$. Writing $\bar{\bar{L}} = (\mu\epsilon - v_p^2)\bar{\bar{I}}_t$ and $M^T = M$, we have

$$J(v_p; f) = \frac{\int_S \frac{1}{\mu\epsilon - v_p^2}\left[\nabla f^* \cdot M \nabla f + \frac{1}{v_p} \nabla f^* \cdot J u_z \times \nabla f\right] dS}{\int_S f^* M f \, dS}$$

$$= \frac{\int_S \frac{1}{\mu\epsilon - v_p^2}\left[\epsilon |\nabla e|^2 + \mu |\nabla h|^2 + \frac{2}{v_p} \Re\{u_z \cdot \nabla e \times \nabla h^*\}\right] dS}{\int_S (\epsilon |e|^2 + \mu |\mu|^2) \, dS}. \quad (41)$$

This expression can be seen to coincide with that given in [10] (note that the factor $1/v_p$ in the (21) of the referenced paper is erroneously missing).

III. Discussion

The functional (39) is exact and presents an obvious extension of previously known functionals from isotropic to bi-isotropic media. It can be applied to obtaining approximate mode solutions for open bi-isotropic waveguides. By approximating the longitudinal field functions, the dispersion function $\beta(\omega)$ can be obtained point by point. In practice, this can be performed using the following scheme:

1. Choose a value for the phase velocity parameter v_p.
2. Find suitable approximating functions for the longitudinal fields $e(\boldsymbol{\rho})$ and $h(\boldsymbol{\rho})$, i.e., the matrix $f(\boldsymbol{\rho})$, with free parameters. Insight on the field distribution of the problem will help in allowing use of just a few parameters; otherwise, a massive computation scheme with a great number of parameters is needed.
3. Optimize values of these parameters so that the functional $J(v_p; f)$ obtains the stationary value; i.e., its differentiation with respect to all these parameters is zero. (In case of large number of parameters this requires use of some optimization procedure.)
4. The corresponding parameter values inserted in the longitudinal field expressions give closest approximations to the fields and the functional value approximates the value of ω^2.
5. Now it is easy to determine a point on the dispersion diagram: $\omega = \sqrt{\omega^2}$, $\beta = \omega v_p$.
6. The transverse field functions corresponding to this point are obtained from (23).
7. To find another point, start with another value of v_p.

Thus, the procedure works best when some *a priori* knowledge of the longitudinal fields exists, which helps in finding suitable approximating functions with not too many optimizable parameters. Obviously, the method is especially attractive for finding the lowest-order modes with least spatial variation. To find the knowledge required, it appears necessary to work through some examples with brute-force technique. This is, however, outside the scope of the present theoretical study.

References

[1] D. L. Jaggard, J. C. Liu, and X. Sun, "Spherical Chiroshield," *Electron. Lett.*, vol. 27, no. 1, pp. 77–79, Jan. 1991.
[2] I. V. Lindell, A. H. Sihvola, A. J. Viitanen, and S. A. Tretyakov, "Geometrical optics in inhomogeneous chiral media with applications to polarization correction in inhomogeneous lens antennas," *J. Electromagn. Waves Appl.*, vol. 4, no. 6, pp. 533–548, 1990.
[3] N. Engheta and P. Pelet, "Reduction of surface waves in chirostrip antennas," *Electron. Lett.*, vol. 27, no. 1, pp. 5–7, Jan. 1991.
[4] A. Lakhtakia, V. K. Varadan, and V. V. Varadan, *Time-Harmonic Electromagnetic Fields in Chiral Media.* Berlin: Springer, 1989.
[5] B. D. H. Tellegen, "The Gyrator, a new electric network element," *Philips Res. Rep.*, vol. 3, pp. 81–101, 1948.
[6] J. C. Monzon, "Radiation and scattering in homogeneous general biisotropic regions," *IEEE Trans. Antennas Propagat.*, vol. 38, no. 2, pp. 227–235, Feb. 1990.
[7] I. V. Lindell and A. J. Viitanen, "Duality transformations for general bi-isotropic (nonreciprocal chiral) medium," *IEEE Trans. Antennas Propagat.*, vol. 40, no. 1, Jan. 1992.
[8] I. V. Lindell, "Variational methods for nonstandard eigenvalue problems in waveguide and resonator analysis," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-30, no. 8, 1194–1204, August, 1982.
[9] I. V. Lindell and A. H. Sihvola, "Dielectrically loaded corrugated waveguide: variational analysis of a nonstandard eigenproblem," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-31, no. 7, pp. 520–526, July 1983.
[10] I. V. Lindell and M. I. Oksanen, "Transversely anisotropic optical fibers: variational analysis of a nonstandard eigenproblem," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-31, no. 9, pp. 736–745, September 1983.
[11] W. X. Zhang, *Engineering Electromagnetism: Functional Methods.* London: Ellis Horwood, 1991.

Analysis of Bilateral Fin-Lines on Anisotropic Substrates

Thinh Quoc Ho and Benjamin Beker

***Abstract*—A full-wave analysis of the bilateral fin-line on anisotropic substrates is presented. The supporting medium is characterized simultaneously by both nondiagonal second rank $[\epsilon]$ and $[\mu]$ tensors. The dyadic Green's function is formed rigorously in the discrete Fourier transformed domain and is used to study the propagation characteristics of the fin-line. The Green's function elements are given explicitly in their closed forms along with the verification of the theory. New data describing the dispersion properties as functions of the coordinate misalignment are also generated for several substrate materials.**

I. Introduction

Although the theories of transmission lines on anisotropic structures are well documented, the major effort thus far has been di-

Manuscript received June 3, 1991; revised October 4, 1991.
The authors are with the Department of Electrical and Computer Engineering, University of South Carolina, Columbia, SC 29208.
IEEE Log Number 9104773.